Short Note on Correlation in a Hull-White Tree

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Abstract—**Two-factor Hull-White/G2++ trinomial trees can reproduce the continuous model's correlation structure by using a high discretization resolution, or by tweaking the transition probabilities. This paper investigates the approaches' correlation error and gives minimal discretization resolutions that observe some error limit. The results help make value-at-risk calculations more efficient.**

Keywords: Hull-White, G2++, discretization resolution, correlation

1 Handling Discrete Transitions

Given the usual two-factor Hull-White [2] dynamics in a G2++ notation

$$\begin{aligned} r(t) &= x(t) + y(t) + \phi(t), \quad r(0) = r_0, \\ x(t) &= -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0, \\ y(t) &= -by(t)dt + \eta dW_2(t), \quad y(0) = 0, \end{aligned}$$

$$dW_1(t)dW_2(t) = \rho dt,$$

a trinomial tree implementation [3] must set discrete joint transition probabilities that match the marginal distributions of $x(t)$ and $y(t)$, and lead to a discrete correlation that approximates ρ.

One way to determine the transition probabilities is Brigo and Mercurio's [1] one (*BM*):

1: For every node, determine the marginal transition probabilities and the resulting uncorrelated joint transition probabilities Π^0.

2: Set the transition probabilities Π^ρ to

$$\begin{aligned} \Pi^\rho &= \Pi^0 + \Pi^\Delta, \\ \Pi^\Delta &= \mathbf{1}_{\rho>0}\frac{\rho}{36}\begin{pmatrix} -1 & -4 & 5 \\ -4 & 8 & -4 \\ 5 & -4 & -1 \end{pmatrix} - \mathbf{1}_{\rho<0}\frac{\rho}{36}\begin{pmatrix} 5 & -4 & -1 \\ -4 & 8 & -4 \\ -1 & -4 & 5 \end{pmatrix} \end{aligned}$$

3: If Π^ρ contains negative values, set $\Pi^\rho = \Pi^0$.

This approach "loses" correlation in step 3; the remedy is to increase the discretization resolution.

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One alternative is effective and computationally cheap—discard step 3, and modify step 2 (*BM'*) [2]:

2: Set Π_ρ to:

$$\begin{aligned} \Pi^\rho &= \Pi^0 + \bar{s}\Pi^\Delta. \\ \bar{s} &= \max\{s | s \in [0,1], \Pi^0 + s\Pi^\Delta >_c 0\} \end{aligned}$$

Intuitively, Π^Δ moves—proportinally to ρ—mass towards the matrix diagonals; $\bar{s}\Pi^\Delta$ moves as much of that mass as possible without violating the probability constraints. The computational costs are negligible, because $\bar{s}$ is

$$\bar{s} = \frac{36}{\rho}\begin{cases} \min\{\Pi^0_{ud}, 4\Pi^0_{um}, 4\Pi^0_{md}, 4\Pi^0_{mu}, 4\Pi^0_{dm}, \Pi^0_{du}\}, & \rho > 0 \\ \min\{\Pi^0_{uu}, 4\Pi^0_{um}, 4\Pi^0_{md}, 4\Pi^0_{mu}, 4\Pi^0_{dm}, \Pi^0_{dd}\}, & \rho < 0 \end{cases}$$

In x, y-regions where the mean reversion is weak, it is equivalent to the old approach; in outer regions (i.e., with large x, y), it avoids falling back to the uncorrelated probabilities, and thus a large correlation loss.

Finally, the transition probabilities can be tweaked even further—a minimization or a brute-force approach can determine the probabilities that best match ρ, subject to the marginal distribution constraints (*BF*) [4].

2 Minimal Discretization Resolution

No approach can guarantee to exactly match the desired instantaneous correlation if ρ is large; all must rely on higher discretization resolutions to match the instantaneous, and in turn the terminal, correlation more closely. We can quantify this required increase in the discretization resolution for BM, BM', and BF.

Let I be the number of discrete equispaced time steps, and $\overline{\mathrm{Corr}}_{x,y}(i)$ the tree's discrete terminal correlation at time step i. Let E_t denote the relative error between this discrete correlation and the analytical correlation $\mathrm{Corr}_{x,y}(t)$ at time t:[1]

$$E_t = -\frac{\overline{\mathrm{Corr}}_{x,y}(\lfloor t/\delta \rfloor) - \mathrm{Corr}_{x,y}(t)}{\mathrm{Corr}_{x,y}(t)},$$

$$\mathrm{Corr}_{x,y}(t) = \frac{2\rho\sqrt{ab}(1 - e^{-(a+b)t})}{(a+b)\sqrt{1 - e^{-2at}}\sqrt{1 - e^{-2bt}}},$$

[1] Because step 3 loses correlation, the numerator is negative; we revert the sign for convenience.

E_t has several properties:

- It increases with ρ: BM produces negative, invalid transition probabilities more often; BM' and BF are unable to match higher ρ values closely.
- It increases with t due to the cumultive way correlation is lost in the tree; we thus only examine E_T.
- It increases with the process parameters a, b: higher mean reversion skews the marginal probabilities due to the discretization; the smaller of the outer probabilities can then lead to negative values or mismatches.
- It does not, however, depend on σ and η, because the spatial discretizations of x and y are determined by their variances.

To illustrate the necessary increase in the discretization resolution, let $I^e_{min}(a, b, \rho)$ be the minimal discretization resolution I such that $E_T \leqslant e$, given the process parameters a, b, ρ. Figure 1 gives such (empirically computed) minimal discretization resolutions which guarantee that $E_T < 1\%$ (for $T = 1$ and $\rho = 0.9$, with the BM approach). Figure 2 compares BM, BM' and BF for $\rho = .95$ and three levels of b.

Figure 1: Minimal discretization resolution for BM such that $E_{T=1} < 1\%$, as a function of a and b ($\rho = 0.9$).

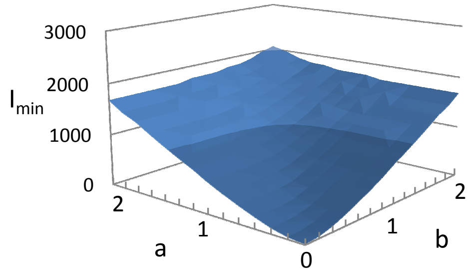

Figure 2: Minimal discretization resolution for BM, BM', BF such that $E_{T=1} < 1\%$, as a function of a ($\rho = 0.95$).

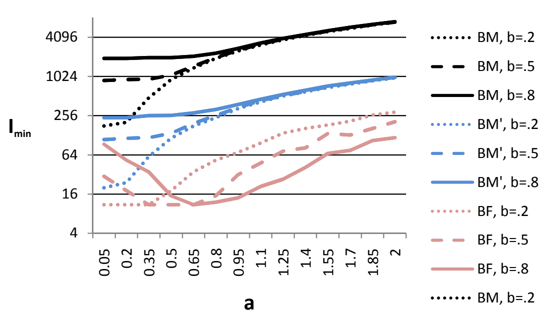

Table 1 gives approximate minimal discretization resolution for BM, BM', BF and various e, ρ, obtained via empirical search for I_{min} and OLS regression. Note: $T = 1$ is used for these approximations; to obtain I_{min} for other T, one can rescale the process parameters a, b, ρ before using the regressions. Clearly, even relatively moderate values for ρ, a, b require high discretization resolutions in case of BM; BM' is preferable. BF further reduces the necessary discretization resolution, but at a higher computational cost; it is an alternative when a tree setup becomes memory-bound.

Table 1: Approximate minimal discretization resolution for a, b, ρ and e: $I^{e,\rho}_{min}(a, b) \approx int + c_a a + c_b b + c_{ab} ab + c_{a^2} a^2 + c_{b^2} b^2$. WLOG, $a < b$. $T = 1$, $a, b \leqslant 2$.

			$e = 2\%$	$e = 1\%$	$e = .5\%$
BM:	$\rho = .9$	int	-110.85	-138.70	-164.07
		c_a	135.85	89.52	140.90
		c_b	484.01	666.02	777.95
		c_{ab}	-151.77	-111.76	-162.40
		c_{a^2}	158.93	154.99	173.78
		c_{b^2}	142.43	131.62	172.23
	$\rho = .95$	int	-463.59	-571.17	-670.40
		c_a	375.01	369.93	402.28
		c_b	2142.73	2746.27	3290.64
		c_{ab}	-469.65	-500.40	-525.16
		c_{a^2}	587.01	635.99	641.48
		c_{b^2}	499.75	554.13	661.26
BM':	$\rho = .9$	int	-15.47	-27.26	-41.67
		c_a	20.39	29.04	38.26
		c_b	73.23	123.88	188.64
		c_{ab}	-18.53	-29.34	-38.39
		c_{a^2}	29.09	41.88	52.53
		c_{b^2}	25.81	38.48	52.54
	$\rho = .95$	int	-42.68	-75.41	-124.41
		c_a	43.46	61.03	99.58
		c_b	178.64	329.04	546.68
		c_{ab}	-41.15	-70.10	-103.20
		c_{a^2}	63.74	104.52	138.34
		c_{b^2}	59.69	103.12	154.79
BF:	$\rho = .9$	int	11.88	8.28	5.71
		c_a	1.49	-11.77	-20.06
		c_b	-6.61	9.05	21.18
		c_{ab}	-35.12	-45.23	-69.11
		c_{a^2}	23.85	33.76	48.87
		c_{b^2}	16.08	18.15	27.10
	$\rho = .95$	int	8.27	1.59	-5.79
		c_a	-31.34	-69.41	-148.79
		c_b	21.61	69.17	152.67
		c_{ab}	-90.78	-181.58	-290.69
		c_{a^2}	67.72	135.71	212.79
		c_{b^2}	33.78	59.93	97.57

Finally, the regressions help minimze the discretization resolutions and thus the computational costs for the main task in a financial organization's daily value-at-risk calculation: pricing large product portfolios under different interest rate scenarios and thus Hull-White setups.

References

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